ON CHERN-SIMONS THEORY WITH AN INHOMOGENEOUS GAUGE GROUP AND BF THEORY KNOT INVARIANTS

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Abstract. We study the Chern-Simons topological quantum field theory with an inhomogeneous gauge group, a non-semi-simple group obtained from a semi-simple one by taking its semi-direct product with its Lie algebra. We find that the standard knot observable (i.e. trace of the holonomy along the knot) essentially vanishes, and yet, the non-semi-simplicity of the gauge group allows us to consider a class of un-orthodox observables which breaks gauge invariance at one point and leads to a non-trivial theory on long knots in $\mathbb{R}^3$.
We have two main morals :

1. In the non-semi-simple case there is more to observe in Chern-Simons theory! There might be other interesting non semi-simple gauge groups to study in this context beyond our example.

2. In the case of an inhomogeneous gauge group, we find that Chern-Simons theory with the un-orthodox observable is actually the same as 3D BF theory with the Cattaneo-Cotta-Ramusino-Martellini knot observable. This leads to a simplification of their results and enables us to generalize and solve a problem they posed regarding the relation between BF theory and the Alexander-Conway polynomial. We prove that the most general knot invariant coming from pure BF topological quantum field theory is in the algebra generated by the coefficients of the Alexander-Conway polynomial.

1. **Introduction**

We would like to address the question of the most general knot invariant coming from BF topological quantum field theory (TQFT). In the mid-90's, Cattaneo et al. [9, 10, 11, 12] showed that while BF theory with cosmological constant produces the same invariants of knots as the Chern-Simons (CS) theory, the BF theory with no cosmological constant (pure BF theory) and SU(2) gauge group produces invariants that lie in the algebra generated by the coefficients of the Alexander-Conway polynomial. The BF TQFT is completely equivalent to CS theory, however while the equivalence with non-zero cosmological constant maintains the semi-simplicity property of the gauge group, the equivalence when the cosmological constant is set to zero shifts us to a CS theory with a non-semi-simple gauge group. Thus, starting with a gauge group G in the pure BF theory we end up on the CS side with a semi-direct product of the group with its Lie algebra $G \ltimes \mathcal{G}$, also known as the inhomogeneous group of G (denoted IG). Eventually, the question we want to address is as follows:

What is the most general knot invariant (or knot observable) in a CS theory with an inhomogeneous gauge group ?

The natural thing to begin with is the standard gauge invariant observable: the trace of the holonomy of the gauge connection along the knot, in some chosen representation. As we shall see in chapter 2, this observable gives us no information regarding the knot (except for framing information that can be normalized to zero anyway). This fact seems to pose a contradiction since in the equivalent pure BF theory one can extract some non-trivial information regarding the knot (at least in the case of SU(2) gauge). Therefore, there must be a procedure that will give us non-trivial information regarding the knot in this setting.

In this paper we will introduce the following procedure (section 3.1): We take an observable which is not gauge invariant at one single point on the knot (holonomy along the knot without the trace). By doing that, we get a CS theory on S^3 with broken gauge symmetry at one point on the knot. Then, we declare this point to be infinity by taking the point out of our space (i.e. puncturing S^3 at that point). Since the gauge transformations are taken to vanish at infinity, we get a completely invariant CS theory on $\mathbb{R}^3$ with a long knot embedded in it. Since knot theory in S^3 and long knot theory in $\mathbb{R}^3$ are "isomorphic" theories we lose no information as far as knot theory is concerned and we get "legal" CS theory in which we can consider a new and wider class of observables.

Using perturbation theory, we will show that our observable gives non-trivial information about the knot (section 3.2). Continuing with perturbation theory in our setting (CS with IG gauge group and the new observable) we will be able to prove that the most general knot invariant coming from such a construction is in the algebra generated by the coefficients of the Alexander-Conway polynomial (section 3.3). Returning to BF theory (chapter 4) we will create an observable that reproduces Cattaneo et al.'s result for SU(2) and generalizes it to every metrizable gauge group and representation thereof used in the construction of the pure BF theory.

Our construction detects, in some sense, the difference between the invariant subspace of the universal enveloping algebra of the non-semi-simple gauge group and the co-invariant

quotient of the universal enveloping algebra. This enables us to extract non-trivial information regarding the knot (a difference which does not exist in the semi-simple case). We discuss this issue in chapter 5.

2. Perturbation theory for the standard observable with an inhomogeneous gauge group

In this section we will show that the perturbation theory of Chern-Simons (CS) theory with an inhomogeneous gauge group and the standard knot observable is almost trivial. We start (2.1) with some reminders about perturbation theory (in CS context). Unfortunately, due to the size of the topic this reminder is not meant to teach the theory. For recent detailed pedagogic reviews see [19, 18]. This will be followed by introduction of the inhomogeneous gauge group and our notations for it (2.2). We continue with giving more details about perturbation theory using this gauge group (2.3). Section 2.3 will describe the consequences of using such a gauge group on the knot invariants coming from the standard observable. Finally, in section 2.4, we will show that the standard construction in this setting is trivial, leading to (almost) trivial knot invariants.

2.1. **Some reminders about perturbation theory and Chern-Simons theory.** We recall the setting for knot invariants in the framework of CS theory (in this chapter a knot is an embedding of S^1 into S^3). We take S^3 to be the 3 dimensional manifold on which the CS theory is defined. We choose any gauge group G whose lie algebra is metrizable (e.g. semi-simple gauge group). Denote the metric on the Lie algebra by $<,>$ and let A be a G-connection. We consider the following action functional:

$$\text{Chern-Simons action } CS(A) = k \cdot 2\pi \int_{S^3} < A \wedge dA + \frac{2}{3} A \wedge A \wedge A >$$

where k is a coupling constant which satisfies a quantization condition (e.g. for semi-simple gauge groups it must be an integer). In what follows, however, we will work perturbatively with formal power series in k^{-1}.

Now, recall the standard knot observable - $Tr_R hol(A)$ where $hol(A)$ stands for the holonomy of the connection A along the knot (the path-ordered exponent of the integral of A along the knot) and the trace is taken in some chosen representation R of the gauge group.

One obtains a knot invariant by taking the path integral over all connections with the observable plugged into the integrand :

$$Z(knot) = \int \mathcal{DA}\ e^{iCS(A)} Tr_R hol(A)$$

This integral is usually referred to as the expectation value of the standard observable and it is viewed as a function from knots to the base field (the complex numbers from now on). One is usually interested in $\frac{Z(knot)}{Z(\emptyset)}$, where $Z(\emptyset)$ is the path integral with no observable plugged in (no knot).

Calculating the integral using perturbation theory we get the following sum:

$$\sum_{D \in D_{CS}} k^{-degD} \cdot \zeta^{CS}(D) \cdot W_{\mathcal{G}}(D)$$

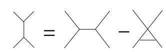
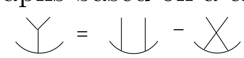
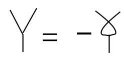
where D_{CS} is defined to be the set of all trivalent connected graphs based on S^1 (Feynman diagrams based on a Wilson loop), ζ^{CS} is the integration of the corresponding propagators over the diagram (that is, the integration over the appropriate configuration space) and $W_{\mathcal{G}}$ is the Lie-algebraic part of the expectation value which holds all the information coming from the Lie group G and its representation R (that is, the weight system, reviewed quickly below).

In the CS perturbation theory calculation, every internal edge contributes $1/k$ and every internal vertex contributes k to the total power of k of the diagram (recall that an internal edge/vertex is an edge/vertex which is not on the Wilson loop). The total power of $1/k$ is called the *degree* of the diagram. A convenient way of counting the degree is labeling each internal vertex with -1 and each edge with +1 and then summing all the labels to get the degree of the diagram. This number is denoted $degD$.

We want to factorize the perturbation theory calculation in order to get a better hold on the above knot invariant. As a first step, we can drop the weight system and replace it with the diagram D on which it is calculated. We get the following invariant:

$$\sum_{D \in D_{CS}} k^{-degD} \cdot \zeta^{CS}(D) \cdot D$$

This invariant is viewed as taking values in $\mathcal{A}(S^1)$, where we define $\mathcal{A}(S^1)$ as follows :

Definition 2.1. $\mathcal{A}(S^1)$ is the algebra of all connected trivalent graphs based on a circle (i.e. D_{CS}) quotient by the IHX relation , the STU relation  and the anti-symmetry relation .

When we apply the weight system $W_{\mathcal{G}}$ to this invariant we get the first sum. Next we will apply the weight system by parts, in order to get a further factorization.

Weight systems are described and explored in [6]. We recall the main construction of a weight system coming from a Lie algebra, which includes three parts: labeling, contracting and the trace part. In the labeling part, after choosing a basis for the Lie algebra, one labels each internal edge of a given diagram (i.e. edges that are not contained in the base circle) with a different index on each end of the edge. In the second part, one writes the structure constants tensor for each internal vertex and the metric tensor for each internal edge (using the indices on these vertices/edges and the chosen basis) and proceeds to contract matching indices (using the metric to raise and lower indices). At that point, one is left with an invariant tensor in the enveloping algebra of $\mathcal{G}$ whose indices are the ones on the vertices on the base loop. In the final part, the trace part, one represents this tensor using a chosen representation of $\mathcal{G}$ and takes the trace. This concludes the construction of the weight system.

Taking a diagram (more precisely a class representative but we will not make these distinctions throughout the paper) in $\mathcal{A}(S^1)$ and applying to it the labeling and contracting parts of the weight system gives us a tensor in the universal enveloping algebra. This tensor is invariant under the adjoint action (all the tensors used in building it are invariant under the action and contractions are $\mathcal{G}$-maps) and therefore it is in the invariant subspace of the universal enveloping algebra $U(\mathcal{G})^{\mathcal{G}}$. Although we did not apply the trace part of the weight system yet, we still have a trace of the trace that appears in the standard observable we

started with, which comes about diagrammatically in the fact that we are looking at diagrams based on S^1. The ability to move legs around cyclically (which is the trace property) forces us to quotient the resulting tensor in $U(\mathcal{G})^{\mathcal{G}}$ into the co-invariant quotient of the algebra, thus getting an invariant in $(U(\mathcal{G})^{\mathcal{G}})_{\mathcal{G}}$. Finally, one can apply the trace part to get elements of $\mathbb{C}$.

To summarize, when we use perturbation theory for the standard observable, we get a knot invariant in the form of a series (in $1/k$) with coefficients in $\mathbb{C}$, which factorize in the following way:

$$\{knots\} \to \mathcal{A}(S^1)[[k^{-1}]] \to (U(\mathcal{G})^{\mathcal{G}})_{\mathcal{G}}[[k^{-1}]] \to \mathbb{C}[[k^{-1}]] \tag{1}$$

Although there is much more to say about Chern-Simons theory and the use of perturbation theory in this context, we will stop at this point. We recommend [4, 5, 3, 2, 1, 15] for various approaches (physically and/or mathematically inclined) in addition to the reviews cited above. The fact that we actually get a knot invariant in every step of the factorization above is an example of an important issue which we completely ignored here.

2.2. **The inhomogeneous gauge group.** We now present a specific type of gauge group that we wish to focus on in the context of perturbation theory of CS theory.

Start with a semi-simple gauge group G and take a semi-direct product of it with its Lie algebra (the semi-direct action is the adjoint action). Look at the Lie algebra of this semi-direct product and denote it L_0. As a vector space, L_0 is a double copy of the original algebra, $\mathcal{G} \bigoplus \overline{\mathcal{G}}$, where we use upper bar notation to distinguish the second copy. For any $X \in \mathcal{G}$ let $X \mapsto \overline{X}$ be the identity map between the two copies. Thus $\overline{X}$ is the element X in the second copy of $\mathcal{G}$. Let X_i be a basis for $\mathcal{G}$, $\overline{X_i}$ the corresponding basis for $\overline{\mathcal{G}}$ and [.,.] the original bracket structure on $\mathcal{G}$. We have the following bracket structure on L_0:

$$[X_i, X_j]_{L_0} = [X_i, X_j] \qquad [X_i, \overline{X_j}]_{L_0} = \overline{[X_i, X_j]} \qquad [\overline{X_i}, \overline{X_j}]_{L_0} = 0$$

We see now that the second copy is the abelianization of the original algebra.

Letting the metric on $\mathcal{G}$ be $<,>$ (the one coming from the trace of the adjoint representation, say) we take the following (invariant) metric on L_0:

$$< X_i, X_j >_{L_0} = 0 = < \overline{X_i}, \overline{X_j} >_{L_0} \qquad < X_i, \overline{X_j} >_{L_0} = < X_i, X_j >$$

2.3. **The structure of the perturbation theory expansion with the inhomogeneous gauge group.** Let us look at the perturbation theory expansion using the inhomogeneous gauge group L_0. As we will show, the factorization (1) factors even further as the weight system using this type of Lie algebra is more refined. We will define the space of directed "legal" diagrams and construct a factorization of the weight system W_{L_0} through a map into it.

As known from standard perturbation theory of quantum gauge theories, the algebraic contributions of vertices and edges in a Feynman diagram are determined by the structure constants and the metric on the Lie algebra. These contributions are encoded in the weight system, as described earlier.

Given a diagram, choose a basis X_i to $\mathcal{G}$, take the corresponding basis $\overline{X_i}$ to $\overline{\mathcal{G}}$ (obtaining a basis to the entire algebra) and apply the weight system using the following steps:

1(pre). Sum over all the ways of labeling the ends of the edges of the diagram with different indices i and $\bar{i}$ representing the basis for $\mathcal{G}$ and $\overline{\mathcal{G}}$ respectively. Therefore, instead of labeling with one type of indices running over the entire basis of the algebra, label with two types of indices, each running over one summand in the direct sum that forms the algebra.

2(pre). Write the structure constants tensor and the metric tensor (as before) with indices according to the labeling, and contract the matching indices (thus in each summand of step 1(pre) only a part of the algebra basis is contracted).

Since the summations are finite, one can see that this way of applying the labeling and contraction parts of the weight system (i.e. in two summation steps having two types of algebra indices instead of one) is equivalent to the usual way of applying the weight system.

Furthermore, we know that certain combinations of indices vanish in the second step. By looking at the metric we observe that the only edges which are non zero are the ones that have a non-bar index on one end and a bar-index at the other end. By looking at the bracket structure we observe that the only combination of indices on a vertex that will not be zero at step 2(pre) is a combination of two non-bar indices and one bar index. This allows us to revise step 1(pre) above into:

1(revised). Sum over all ways of labeling the ends of the edges of the diagram with indices i and $\bar{i}$, representing the basis for $\mathcal{G}$ and $\overline{\mathcal{G}}$ respectively, in such a way that each edge has one bar index and one non-bar index at its ends, and each vertex has two non-bar indices and one bar index at its legs.

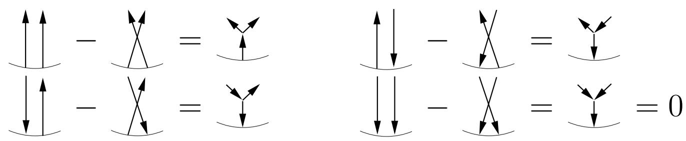
Let us introduce a notation. Wherever we have an edge after step 1(revised) we direct the edge (by putting an arrow head) from the non-bar index to the bar index and drop the bar on top of the bar index. Thus the two steps for applying the weight system of L_0 on a diagram can be finalized into the following form :

1. Sum over all ways of directing the edges of the diagram (internal edges only) in such a way that each vertex has two outgoing legs and one ingoing leg (we call it a "legal" directing and it looks like ).

2. Put a different index on each end of all the directed edges, write down the appropriate tensors and contract indices according to the arrow convention above (i.e. the arrow's direction indicates if the index is considered bar or non-bar, and we use the appropriate basis accordingly).

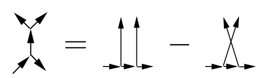
Step 1 is a map from the space of all Feynman diagrams D^{CS} to the space of all directed Feynman diagrams $\overrightarrow{D^{CS}}$, defined to be all trivalent connected graphs based on a circle with arrows on the internal edges (the edges which are not a part of the base circle) such that each vertex is “legal” in the above sense. This map takes a diagram to the sum of all possible “legal” directing of it. In order to extend this map to $\mathcal{A}(S^1)$ we need to treat the STU/IHX/AS relations. For that purpose we define the directed STU, directed IHX and directed AS relations which we will use in taking a quotient of $\overrightarrow{D^{CS}}$. These relations reflect the algebraic structure of L_0 and can be read directly from the bracket structure.

Definition 2.2. The Directed STU relations in $\overrightarrow{D^{CS}}$ are the following relations :

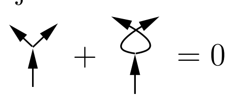


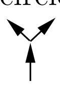
Where the bottom non-directed arc is a part of the base circle.

The Directed IHX relations are actually a consequence of the directed STU relations in $\overrightarrow{D^{CS}}$, just as in [6]. We draw one example here and the rest are just rotations thereof :



The Directed AS relation is defined just as the usual AS relation :

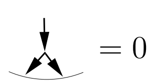


Definition 2.3. The space of all “legally” directed diagrams modulo the directed STU/IHX and AS relations, denoted $\overrightarrow{\mathcal{A}(S^1)}$, is the space of all trivalent connected graphs based on a circle (the Wilson loop) with arrows on the internal edges such that each vertex looks like  (i.e. the space $\overrightarrow{D_{CS}}$ defined above), quotient by the directed STU, directed IHX and directed AS relations defined above.

Armed with these definitions we can see that step 1 of the factorization above is well defined as a map $\mathcal{A}(S^1) \longrightarrow \overrightarrow{\mathcal{A}(S^1)}$. Composed with step 2 above (labeling and contracting of indices) and with taking the trace of the resulting tensor in some chosen representation, we get the following factorization of the perturbation theory expansion of CS theory with an inhomogeneous gauge group and the standard observable :

$$\{knots\} \longrightarrow \mathcal{A}(S^1)[[k^{-1}]] \longrightarrow \overrightarrow{\mathcal{A}(S^1)}[[k^{-1}]] \longrightarrow (U(L_0)^{L_0})_{L_0}[[k^{-1}]] \longrightarrow \mathbb{C}[[k^{-1}]] \tag{2}$$

Note that the directed STU relations (which reflect the structure of L_0) tell us that two adjacent legs which touch the base circle and have arrows pointed towards it, are commutative (as the bar-indices are commutative). On the other hand, if they originate from one vertex they are anti-commutative according to the AS relation. Thus one gets an important relation in $\overrightarrow{\mathcal{A}(S^1)}$:

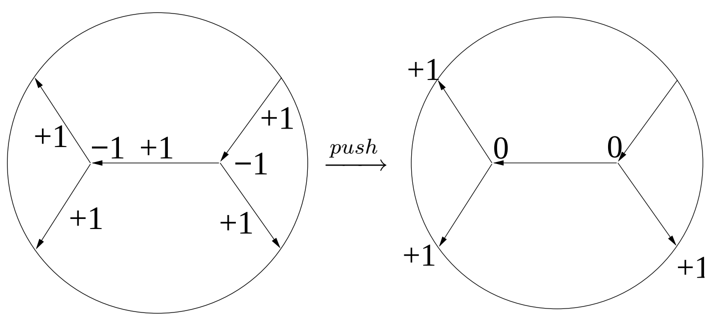


This relation will be used in the arguments presented in the next section.

2.4. **But $\vec{\mathcal{A}}(S^1)$ is (almost) empty!** We will now prove that there are only few non-zero diagrams in $\overrightarrow{\mathcal{A}(S^1)}$ and the ones that are not zero encode a very specific type of information regarding the knot — the framing. This information can actually be normalized to zero showing that the entire weight system W_{L_0} is trivial. We will prove that the only primitive element which is non-zero in $\overrightarrow{\mathcal{A}(S^1)}$ is the directed single chord diagram  (drawn without the arrow here) and thus the only degree n diagram which is non-zero in $\overrightarrow{\mathcal{A}(S^1)}$ is ^{n}. We will do that by first showing that the only primitive elements that are possibly non-zero are the wheel diagrams (e.g. , ) and . We will then present an argument as to why the wheel diagrams are actually zero. We remind the reader that whenever we mention a diagram we mean a (representative of) diagram class.

Lemma 2.4. *Every degree n diagram in $\overrightarrow{\mathcal{A}(S^1)}$ has at least n external vertices (vertices that are on the base loop).*

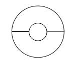
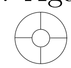
Proof. As already mentioned, to every diagram we attach a power of $1/k$ which is called the degree. A convenient way of counting the degree is labeling each internal vertex with -1 and each edge with +1 and then summing up all the labels to get the degree of the diagram. Given a degree n diagram in $\overrightarrow{\mathcal{A}(S^1)}$, label it with +1 and -1 according to the above. "Push" the labels on the edges in the direction of the arrows. Whenever a label hits a vertex it stops and the labels at the vertex are added. An example of this procedure for a degree 3 diagram is given by:



According to the "legal" directing rules, each internal vertex will have one label +1 hitting the vertex label -1 thus summing to a zero label. All other labels are pushed to the external vertices (on the Wilson loop). Since the total sum of labels is conserved and equal to the degree of the diagram, what we have just proven is that a degree n diagram in $\overrightarrow{\mathcal{A}(S^1)}$ has at least n vertices on the Wilson loop (in other words at least n external legs).

This lemma has immediate consequences in terms of the primitive elements that generate the algebra $\overrightarrow{\mathcal{A}(S^1)}$. The primitive elements of the non-directed algebra $\mathcal{A}(S^1)$ are these diagrams which remain connected when the Wilson loop is removed from them. Let $\mathcal{P}_n$ be the space of primitive diagrams in $\mathcal{A}(S^1)$ of degree n. Filter the primitive spaces (of different degrees) according to the number of external legs and write $\mathcal{P}_{n,d}$ for the space of primitive diagrams in $\mathcal{A}(S^1)$ of degree n and at most d external legs.

Lemma 2.5. [13, 14] *If n is even, then $\mathcal{P}_n = \mathcal{P}_{n,n}$ and the quotient space $\mathcal{P}_{n,n}/\mathcal{P}_{n,n-1}$ is one dimensional generated by the n-degree wheel. If n is odd, then $\mathcal{P}_n = \mathcal{P}_{n,n-1}$ (with the exception of $\mathcal{P}_1$ being generated by $\ominus$)*

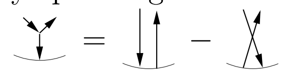
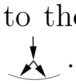
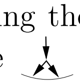
Proof. See [13, 14, 6] for proof and much more details on the subject. Again, we mention that a wheel diagram looks like that: the two wheel  , the 4-wheel  and so on.

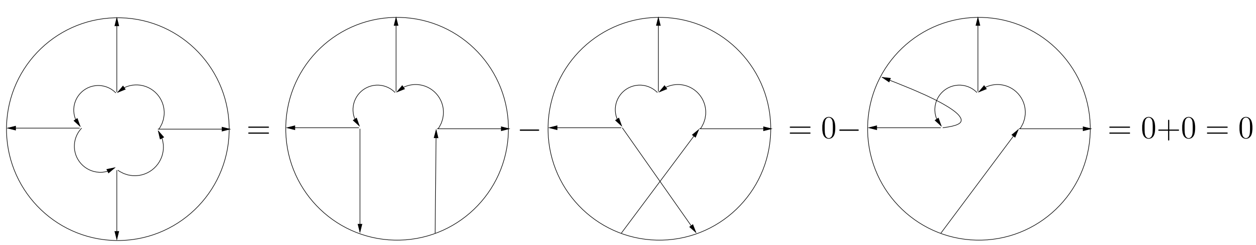
This lemma transfers over to $\overrightarrow{\mathcal{A}(S^1)}$ where the primitives are the primitives of $\mathcal{A}(S^1)$ directed in all possible ways. Combining the previous two lemmas we get:

Corollary 2.6. *The only primitive diagrams that might possibly be non zero are the wheel diagrams and $\ominus$.*

However,

Lemma 2.7. *The wheel diagrams are zero in $\overrightarrow{\mathcal{A}(S^1)}$.*

Proof. We start by looking at a directed wheel diagram. There are only two directed wheel diagrams of each degree, one for each of the two ways of directing the inner loop, and it does not matter which one we choose to look at. Due to the "legal" directing rules, it is not hard to see that the external legs are always pointing towards the base loop. We apply a directed STU relation on one of these legs  obtaining two tree diagrams (i.e. diagrams which contain no internal loops). A tree diagram will always have exactly one leg pointing from the base loop inwards. The tree diagram that corresponds to the first summand on the right hand side of the relation has a vertex which looks like . This means the tree diagram is equal to zero (see the comment at the end of section 2.3). The second tree diagram, corresponding to the second summand on the right hand side of the directed STU relation, is "leg crossed". This crossing can be untangled by moving the crossed leg around the base circle using the directed AS relation when necessary and the commutativity of outward pointing legs. Moving the leg around the circle will yield a diagram which again has a vertex which looks like  (up to a sign) and thus equals to zero. Apply this argument to any n-wheel diagram and the lemma is proven.
We demonstrate the entire argument on the 4-wheel :



Combining all of the above lemmas we finally get :

Proposition 2.8. *The only possible non zero primitive element in $\overrightarrow{\mathcal{A}(S^1)}$ is the directed single-chord diagram $\ominus$. Thus, the only diagrams of degree n that are possibly non zero in $\overrightarrow{\mathcal{A}(S^1)}$ are directed $\ominus^n$.*

Proof. The proof is a corollary of all the above lemmas. The only primitive element which is non-zero is $\ominus$ and it generates powers of itself and sums of such powers.

In CS perturbation theory it is well known that the single-chord diagram has the framing (self-linking) of the knot as its integral $\zeta^{CS}(\ominus)$ (see for example [15]). Higher degree diagrams, which are merely power of the single-chord diagram, will hold information encoded in powers of the framing of the knot. Moreover, this framing number can be normalized as we wish, including normalization to zero, thus we can summarize this chapter with the following proposition:

Proposition 2.9. *CS topological quantum field theory with an inhomogeneous gauge group and the standard knot observable (expectation value of the trace, in some representation, of the holonomy along the knot) holds no more than the framing information (which can be trivialized anyway).*

CS theory with an inhomogeneous gauge group and the standard observable fail to recognize knotting in S^3. As a gauge theory, CS theory with inhomogeneous gauge group is equivalent to pure BF theory and should therefore "see" knots at least for ISU(2) gauge group, as was shown by Cattaneo et al. [9, 10, 11]. Therefore, we need a procedure that will extract some non trivial information regarding the knot in our setting. This is what we will be presenting in the next chapter.

3. Extracting non trivial information regarding the knot

3.1. **Breaking gauge symmetry at one point - puncturing the space.** As seen in the previous chapter, the reason we got zero information when the standard observable is used is the "emptiness" of $\overrightarrow{\mathcal{A}(S^1)}$. Working with the inhomogeneous gauge group allows us to factor through the "legally" directed diagrams space when applying perturbation theory thus there is no way of avoiding the consequences of lemmas 2.3 and 2.4 of the previous chapter which state that the only possible non-vanishing contribution in perturbation theory comes from the wheel diagrams. The best we can do is try and avoid the reasons lemma 2.5 is true. When we look at the proof of lemma 2.5 we can realize that the cause for the vanishing contribution of the wheel diagrams is the fact that legs can be moved around the Wilson loop (the base circle of the diagram). Dealing with the loop translates (observable-wise) to dealing with the trace (and its cyclic properties) in the standard observable. In diagrammatic terms (perturbation-wise) this means trying to cut the base circle open.

Assume first that we just remove the trace from the standard observable of a knot in S^3. We get an expectation value which is formally a tensor in the universal enveloping algebra $\int \mathcal{D}\mathcal{A}\, e^{iCS(A)} hol \in U(L_0)$. This expression, though, is not well defined since the holonomy by itself is not well defined and transforms non-trivially by conjugation under gauge transformation $\chi(x)$. Choosing (in advance) one point on the knot, say x_0, the conjugation is done by the gauge element at that point ($hol \to \chi(x_0) hol \chi^{-1}(x_0)$). The remedy for that would be to view this (not well defined) invariant in the co-invariant quotient $U(L_0)_{L_0}$. The way this is done (observable-wise) is of course by bringing back the trace. We have done nothing then.

This leads us to try and have special considerations for x_0 (the chosen point on the knot) and the gauge symmetry at that point. Let us break the gauge symmetry at x_0. There are

a few equivalent ways of looking at that process. The first is by declaring that the gauge transformation at x_0 is always the identity element. This might seem to be a bit artificial but an equivalent way of doing it is taking x_0 out of the space on which the theory is defined by puncturing S^3 on x_0. In other words, "send" x_0 to infinity where gauge transformations are taken to vanish. This shifts us to a theory on a long knot (embedding of $\mathbb{R}$) in $\mathbb{R}^3$ with an invariant $\int \mathcal{D}\mathcal{A}\ e^{iCS(A)} hol \in U(L_0)$ which is well defined in the framework of CS theory.

To summarize, we move from a CS theory on S^3 and a knot (embedding of S^1) in it to a CS theory on $\mathbb{R}^3$ and a long knot (embedding of $\mathbb{R}$) in it. This is done by choosing a special point on the knot, puncturing the space there and declaring the puncture as infinity. The standard observable $Tr_R hol$ is replaced by a more general observable hol.

As far as knots are concerned, "regular" knots and long knots are equivalent and we do not lose any information regarding the knot in the process above. Moreover, *hol* (and functions of it) is actually the most general observable in this framework since the connection can always be recovered (up to gauge transformations) from the holonomy information. Thus we work in the setting which allows us to extract maximum information regarding the knot when using the inhomogeneous gauge group.

We take now $\int \mathcal{D}\mathcal{A}\ e^{iCS(A)} hol$ and apply the same perturbation theory factorization as in chapter 2. There are several differences to note:

First, we work with a long knot. Thus, all the Feynman diagrams are not based on a loop but on an interval (representing the embedding of our long knot $\mathbb{R}$). The other properties of the diagrams do not change, though. We define :

Definition 3.1. $\mathcal{A}(I)$ is the algebra of all connected trivalent graphs based on an interval, modulo the STU, IHX and AS relations. (i.e. $\mathcal{A}(S^1)$ with the circle replaced by an interval). $\overrightarrow{\mathcal{A}(I)}$ is defined the same way as $\overrightarrow{\mathcal{A}(S^1)}$, except that the diagrams are based on an interval and not on a circle.

Second, the knot invariant factors through the invariant subspace of the universal enveloping algebra $U(L_0)^{L_0}$. This time there is no necessity in taking the co-invariant quotient of that space, a necessity that came about because of the base circle (or the trace in the observable). Note that the fact that we indeed get a tensor in the invariant subspace tells us that the construction was independent of the choice of the special point x_0.

Third, we do not have a map to $\mathbb{C}$ yet.

All together we get the following factorization for the perturbation theory of CS theory on $\mathbb{R}^3$ with a long knot in it (we denote by l the last map):

$$\{knots\} \to \mathcal{A}(I)[[k^{-1}]] \to \overrightarrow{\mathcal{A}(I)}[[k^{-1}]] \xrightarrow{l} U(L_0)^{L_0}[[k^{-1}]] \tag{3}$$

3.2. **Building an observable and extracting non trivial information.** We have not shown so far that the algebra $\overrightarrow{\mathcal{A}(I)}$ is non-zero and that our construction is indeed non-trivial in the sense that it fixes the problems encountered with the standard observable. We have also not defined a scalar observable, i.e. a map $U(L_0)^{L_0} \to \mathbb{C}$ which completes the perturbation theory factorization. We will now achieve both goals by finding an explicit functional from $\overrightarrow{\mathcal{A}(I)}$ to $\mathbb{C}$ that does not vanish when composed with the above factorization. We will give a formula for a scalar function $U(L_0)^{L_0} \to \mathbb{C}$ that is not zero on the part of $U(L_0)^{L_0}$ which comes from wheel diagrams in $\overrightarrow{\mathcal{A}(I)}$. This observable cannot possibly be any trace of *hol* in some representation, since that will bring us back to the case of chapter 2. Thus we need to find some other scalar function of *hol*.

Start with an n-dimensional representation of $\mathcal{G}$ and denote it B. Define the following representation R of L_0:

$$R(g) = \begin{pmatrix} B(g) & 0 \\ 0 & B(g) \end{pmatrix}, \; g \in \mathcal{G} \qquad\qquad R(\overline{g}) = \begin{pmatrix} 0 & B(g) \\ 0 & 0 \end{pmatrix}, \; \overline{g} \in \overline{\mathcal{G}} \tag{4}$$

It is straight forward to verify that the commutation relation for R is indeed compatible with the commutation relations of the semi-direct product and that R is indeed a representation of L_0. We extend R to $U(L_0)$ in the usual way.

Definition 3.2.

1. Let Δ denote the co-multiplication of the universal enveloping algebra $U(L_0)$ and let Δ^m denote the map $U(L_0) \to U(L_0)^{\otimes m}$ we get by composing $m-1$ times the map Δ.

2. Given the representation R above, denote by $R^{\otimes m}$ the m-tensored representation $U(L_0)^{\otimes m} \to End(\mathbb{C}^{2n})^{\otimes m}$ defined by $u_1 \otimes \ldots \otimes u_m \mapsto R(u_1) \otimes \ldots \otimes R(u_m)$.

3. Define a transformation λ: $End(\mathbb{C}^{2n})^{\otimes m} \to End(\mathbb{C}^{2n})$ by $\tau_1 \otimes \tau_2 \otimes \ldots \otimes \tau_m \longmapsto \tau_1 \cdot C \cdot \tau_2 \cdot C \ldots \cdot \tau_m \cdot C$, where C is the following $2n \times 2n$ matrix: $\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$ and $\cdot$ is matrix multiplication.

4. The "half-trace", denoted $Tr_{\frac{1}{2}}$, is the trace over the upper-left $n \times n$ block of a $2n \times 2n$ matrix:

$$Tr_{\frac{1}{2}} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = Tr(A)$$

Armed with the above definitions and notations we can finally build:

Definition 3.3. Let Σ_m denote the following composition of maps ($\circ$ is used for composition):

$$Tr_{\frac{1}{2}} \circ \lambda \circ R^{\otimes m} \circ \Delta^m : U(L_0)^{L_0} \xrightarrow{\Sigma_m} \mathbb{C}$$

Lemma 3.4. *The map $\Sigma_m \circ l$ is non-zero on the m-wheel diagram for at least one choice of gauge group L_0 and representation R.*

Proof. We start by looking at a directed wheel diagram (i.e. a wheel in $\overrightarrow{\mathcal{A}(I)}$). All the external legs (the ones that touch the base interval) point towards the interval, as we have already seen. This means that after applying l, the tensor w that we get in $U(L_0)$ will have all its components in $\overline{\mathcal{G}}$. After applying Δ^m to w we get a tensor in $U(L_0)^{\otimes m}$ such that each of its summands actually looks like $u_1 \otimes \cdots \otimes u_m \in U(\overline{\mathcal{G}})^{\otimes m} \subset U(L_0)^{\otimes m}$.

Now, the only type of tensors $u_1 \otimes \cdots \otimes u_m \in U(\overline{\mathcal{G}})^{\otimes m}$ that will possibly not result in zero after applying $\lambda \circ R^{\otimes m}$ to it, is the type in which each component u_i is actually an element of $\overline{\mathcal{G}}$ (i.e. tensors of length one in $U(L_0)$ with entry from $\overline{\mathcal{G}}$). This is immediate from the definition of R (4) on $U(\overline{\mathcal{G}})$, the definition of λ and the definition of C.

We summarize then – given a wheel diagram we apply l to it and get a tensor w in $U(L_0)$. $\Delta^m(w)$ is a tensor in $U(L_0)^{\otimes m}$ but the only non zero contributions to $\lambda \circ R^{\otimes m}(\Delta^m(w))$ come from the summands of $\Delta^m(w)$ that look like w or any permutations of its entries :

$$w = l(directed\ m - wheel) = \sum_{i_1,\ldots,i_m=1}^{dim\mathcal{G}} C^{i_1,\ldots,i_m} \overline{X}_{i_1} \otimes \cdots \otimes \overline{X}_{i_m}$$

$$\lambda \circ R^{\otimes m}(\Delta^m(w)) = \lambda \circ R^{\otimes m}\Big(\sum_{i_1,\ldots,i_m=1}^{dim\mathcal{G}} \sum_{\sigma \in S_m} C^{i_1,\ldots,i_m} \overline{X}_{i_{\sigma(1)}} \bigotimes \cdots \bigotimes \overline{X}_{i_{\sigma(m)}} \Big)$$

Where we made a distinction between the tensor product $\otimes$ of $U(L_0)$ and the product $\bigotimes$ of $U(L_0)^{\otimes m}$ for the sake of clarity.

The calculation of $Tr_{\frac{1}{2}} \circ \lambda \circ R^{\otimes m} \circ \Delta^m \circ l(directed\ m - wheel)$ is almost the same as the calculation of the weight system $W_{\mathcal{G},B}$ on the non directed m-wheel diagram (i.e. the directed m-wheel with its arrows forgotten). The only difference is the summation over all permutations of the components of the tensor w above, but that can be done on the diagram level by permuting all external legs of the non directed diagram. Thus we have :

$$Tr_{\frac{1}{2}} \circ \lambda \circ R^{\otimes m} \circ \Delta^m \circ l(directed\ m - wheel) = W_{\mathcal{G},B}(\chi(m - wheel))$$

where $\chi(D)$ is defined to be the sum over all diagrams which differ from D by permutation of the external legs.

In order to show that $\Sigma_m \circ l(directed\ m - wheel)$ is non-zero for at least one choice of L_0 and representation R we let $L_0 = gl(n) \bigoplus \overline{gl(n)}$ with B taken to be the defining representation and calculate the highest order in n of $W_{gl(n),B}(\chi(m - wheel))$. Using simple counting arguments one may show that all the highest order contributions are positive and sum up to a non-zero contribution : $\Sigma_m \circ l(directed\ m - wheel) = mn^{m+1}$ + lower order terms (in n). We leave the exact calculation and arguments to the interested reader. These can be done using exercise 6.36 in [6].

Proposition 3.5. *In $\overrightarrow{\mathcal{A}(I)}$, the wheel diagrams (and possibly the directed $\ominus$) are non zero.*

Proof. We already know that the only possible non-zero directed generators in $\overrightarrow{\mathcal{A}(I)}$ are the wheel diagrams and possibly $\ominus$ (following the exact same arguments for $\overrightarrow{\mathcal{A}(S^1)}$ as in chapter 2). Applying the lemma above for all possible m implies the proposition.

The actual observable: The proposition and lemma above tell us that the perturbation theory (3) for the long knot in $\mathbb{R}^3$ factors through a non-trivial space, and that there is a way of extracting non-trivial information regarding the knot. We can now build an actual quantum observable (function of hol) to plug in the path integral such that the map $Tr_{\frac{1}{2}}\lambda R^{\otimes m}\Delta^m$ will be the last part of the factorization $(U(L_0)^{L_0} \to \mathbb{C})$ in its perturbation theory calculation of the expectation value. This observable is $Tr_{\frac{1}{2}}((R(hol)\cdot C)^m)$, where $R(hol)$ stands for representing the holonomy using the representation R as defined above.

The expectation value $\int \mathcal{D}\mathcal{A}\ e^{iCS}\ Tr_{1/2}((R(hol)\cdot C)^m)$ has a perturbation theory factorization (with Σ_m as the last map):

$$\{knots\} \to \mathcal{A}(I)[[k^{-1}]] \to \overrightarrow{\mathcal{A}(I)}[[k^{-1}]] \xrightarrow{l} U(L_0)^{L_0}[[k^{-1}]] \xrightarrow{\Sigma_m} \mathbb{C}[[k^{-1}]] \tag{5}$$

3.3. **The most general knot invariant in this setting - The Alexander-Conway polynomial.** We have just seen on the last section that we can extract all the information contained in the wheel diagrams by using CS theory with the inhomogeneous gauge group on long knots in $\mathbb{R}^3$ (with the new class of observables). This is opposed to the fact that the standard observable in CS theory with the inhomogeneous gauge group will not see these knots in S^3. We now turn to the task of recognizing the non trivial information we extracted.

Theorem 1. *The most general knot invariant coming from Chern-Simons topological quantum field theory with an inhomogeneous gauge group is non-trivial and is in the algebra generated by the coefficients of the Alexander-Conway polynomial (together with possible framing information).*

Proof. In [17], following motivations from [16, 7], it is proven that any weight system which is zero on the kernel of the de-framing operator and zero on all the primitive spaces which are not wheels, belongs to a knot invariant which is in the algebra generated by the coefficients of the Alexander-Conway polynomial. Since we showed that the only diagrams that contribute to our weight system are the wheel diagrams, and since we can always choose our knot to be of (standard) zero framing, the theorem follows. Other choices of framing will show up through $\ominus$.

Notice that it is possible to take the semi-direct product of G with $\mathcal{G}^*$, the dual of the Lie algebra. We can follow the exact same considerations and get the same results, this time without the need of the metric on $\mathcal{G}$ (just use the natural dual pairing to get a metric on L_0). Therefore our results are true for any Lie algebra $\mathcal{G}$, not necessarily a metrizable one.

4. **Pure BF Theory and the Alexander-Conway polynomial**

Recall that the pure BF theory has the following action $S_{BF} = \int_{S^3} Tr(B \wedge F)$, where F is the curvature of the G-connection A and B is a one form taking values in the algebra $\mathcal{G}$. In [9, 10, 11] Cattaneo et al. took $Tr_R(Exp(\lambda\gamma_1))$ as the most general observable one can consider in Pure BF theory, as long as the standard (zero) framing of the knot is chosen. Here R is any representation of G, γ_1 is $\oint_x Hol(A)B(x)Hol(A)$ (the first degree element in the Taylor expansion of $Hol(A+\kappa B)$ around $\kappa = 0$) and λ is a "coupling" constant counting orders.

Using this observable, QFT techniques and the Melvin-Morton conjecture (MMR, see [7]), it was proven [9] that the set of unframed knot invariants we get from the pure BF theory, using SU(2) gauge group, is generated by the coefficients of the Alexander-Conway polynomial.

In this paper we proved:

Theorem 2. *The most general knot invariant coming from pure BF topological quantum field theory with any gauge group whose Lie algebra is metrizable and with any representation, is in the algebra generated by the coefficients of the Alexander-Conway polynomial.*

Proof. Using the fact that the pure BF theory with gauge group G is just CS theory with gauge group IG [8] this theorem is just a corollary of Theorem 1.

The same generalization for non-metrizable Lie algebras, as right after Theorem 1, applies here as well.

A closer look at our observable $Tr_{\frac{1}{2}}((R(hol) \cdot C)^m)$ shows that it is actually the m^{th} degree in the expansion of the BF observable $Tr_R(Exp(\lambda\gamma_1))$ (up to numerical factors coming from the exponent expansion). The fact that the BF observable also breaks the gauge symmetry of the pure BF theory at one point (it has to be assumed that the special BF gauge symmetry is identity at one point on the knot) was not emphasized much before (though observed of course in [9, 11, 12]). Our setting gives a natural explanation as to why this is indeed the most general observable for BF theory without any need for taking various limits or referring to MMR. We also get a somewhat clearer explanation as to why one can ignore the γ_0 part of the Taylor expansion above.

One can now say that as far as knot theory and 3 dimensional BF topological quantum field theory (with or without cosmological constant) are concerned, there is nothing new beyond Chern-Simons theory, which can reproduce the same knot theoretical results.

5. **Discussion: Non-Semi-Simplicity**

Algebraically, the main difference between our construction and the standard one is getting invariants in the invariant subspace $U(L_0)^{L_0}$ instead of the co-invariant quotient space $(U(L_0)^{L_0})_{L_0}$ of the universal enveloping algebra.

Our construction gives no new extra information if we work with a semi-simple gauge group instead of L_0. This is true because for semi-simple groups, $U(\mathcal{G})^{\mathcal{G}}$ is isomorphic to

$(U(\mathcal{G})^{\mathcal{G}})_{\mathcal{G}} \cong U(\mathcal{G})_{\mathcal{G}}$. We get no new information from applying our procedure and working in $\mathcal{A}(I)$ instead of $\mathcal{A}(S^1)$, since these spaces are isomorphic (the diagram-space way of expressing the previous isomorphism). Our construction seems to detect the difference between the invariant subspace and the co-invariant quotient of the non-semi-simple Lie algebra we have worked with. In other words, it uses the fact that $\overrightarrow{\mathcal{A}(I)}$ is not isomorphic to $\overrightarrow{\mathcal{A}(S^1)}$ (the map $\overrightarrow{\mathcal{A}(I)} \to \overrightarrow{\mathcal{A}(S^1)}$ that closes the base interval into the base circle has a kernel!).

We saw that there is more to observe in Chern-Simons theory when dealing with a non semi-simple gauge group. A question to be raised is whether other non semi-simple gauge groups can hold the same property as the inhomogeneous group - meaning that the above procedure can extract more information about the knot than what was given by the standard observables.

6. **Acknowledgments**

I wish to thank Dror Bar-Natan for his endless support in this research. I also wish to thank Alberto Cattaneo for reading an early version of this paper and giving his comments.

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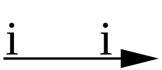
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